jecture E as a conjecture about Haar measure on a certain compact topological group. In this formulation Conjecture E can be thought of as a generalization of Dirichlet's theorem to infinite-dimensional extensions $L$ of $\mathcal{Q}$. For an exposition of this theory, the reader is referred to [2]. If I have said little about methods of proof, it is because there are only a few theorems now proved in the subject. I hope that this talk will generate enough interest to remedy this appalling situation.

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References


MUSIMATICS or THE NUN’S FIDDLE*

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1. The divine ratio. “Abominandum!” said Cicero as he went a purler over a hidden obstacle—“quid est quod?”—and scrabbling in the undergrowth he uncovered an ancient monument. The lettering was illegible but the design—a cylinder circumscribing a sphere—was clearly that which Archimedes, who was killed in the fall of Syracuse 212 B.C., had charged his friends to inscribe on his tombstone. Since Cicero made this discovery about 75 B.C., the tomb has again been lost, probably forever.

Archimedes transformed empirical knowledge into theoretical science and developed the integral calculus which he said would be used by mathematicians “as yet unborn.” In keeping with Aristotle’s dictum that “it is proper to consider the similar even in things far distant from each other,” he considered it highly significant that the cylinder and inscribed sphere, as regards surface

* A symbolic title with Chaucerian overtones. This one-stringed instrument, better known as the ‘Marine Trumpet’, has clarion qualities well suited for trans-Atlantic communication.

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area and volume, are in the ratio 3:2 and that the same relationship exists between the frequencies of an important musical interval.

The ear is very sensitive to this interval—the perfect fifth—and it has been used for tuning instruments from the earliest times. A power of 3/2 can never equal a power of 2/1 and superimposed perfect fifths will never arrive at an octave duplication of the fundamental. On a keyboard instrument, however, we find that twelve fifths pass through the twelve semitones of the chromatic scale and finish on the seventh octave of the fundamental note (Figure 1), which means that the fifths are not all perfect and somewhere the difference between 12 perfect fifths and 7 octaves has been lost. This difference \((3/2)^{12} + 2^7 = 3^{12}/2^{19}\) is called a 'Pythagorean comma.'

![Figure 1](image1)

For tuning purposes the series of fifths is kept within the limits of an octave by descending an octave each time this limit is exceeded (dotted lines Figure 2).

![Figure 2](image2)

The ancient Greeks and Chinese calculated the Pythagorean comma which equals about 24 cents (the cent being one twelve hundredth part of an octave, or the base two logarithm of the ratio multiplied by 1200).

About 40 B.C., King-Fang, a scholar of the Han dynasty, continued the series of superimposed fifths in order to find a closer approximation to an octave. His first improvement came with the 41st fifth which was less than 24 octaves by about 20 cents. Not content, he carried on until he came to the 53rd fifth, which exceeds 31 octaves by about 3.6 cents. This excellent approximation was later recommended by Mercator and the 53 note octave was incorporated in several instruments including Bosanquet's Enharmonic Harmonium which was exhibited in the South Kensington Museum in 1876. In the hope of achieving immortality, I carried on Fang's calculations (without an abacus) and found the next better approximations to be:
Leonardo da Vinci ca. 1470 observed that “two men shouting together do not seem to produce twice the amount of noise that one man would” [1] and in general we now know that sensations vary as the logarithm of the stimulus (Fechner’s Law). We talk and think of two octaves as twice the size of one (as on the piano keyboard), three octaves as three times the size, and so on. Yet the frequencies of these intervals are in the ratio 2:4:8 ... Base two logarithms are therefore naturally suited for musical purposes and were published in 1670, fifty-six years after Napier’s tables [2]. Modern tables are available [3].

2. Lesser divine ratios. Over the centuries musical opinion has been remarkably consistent—

(a) The satisfying intervals are derived from natural harmonics, the frequencies of which are related as the natural number series 1:2:3 ... .

(b) Successive ratios are favoured and are named ‘superparticular.’ They are an infinite series 2/1, 3/2, 4/3, 5/4, ... .

(c) The lower members are pleasing; the higher members tend to harshness and eventually become unacceptable.

(d) Certain ratios, although within the range of acceptable harshness, are regularly rejected, e.g., 7/6, 8/7, 11/10, 12/11, 13/12, 14/13, ... .

There is no obvious reason for this last empirical fact. However, an analysis of a large amount of material discloses that the ear prefers superparticular ratios that are derived from the first three primes, and when other ratios are omitted we are left with the following finite series of well-known intervals:

\[
\begin{align*}
2/1 & \text{ octave} & 9/8 & \text{ major tone} \\
3/2 & \text{ perfect fifth} & 10/9 & \text{ lesser tone} \\
4/3 & \text{ perfect fourth} & 16/15 & \text{ diatonic semitone} \\
5/4 & \text{ major third} & 25/24 & \text{ chromatic semitone} \\
6/5 & \text{ minor third} & 81/80 & \text{ comma of Didymus}.
\end{align*}
\]

The enthusiast will no doubt relate these intervals (excluding the octave) to the nine Platonic and Kepler-Poinsot regular polyhedra. Since the perfect fifth and the major third contain the first three primes, all other intervals may be compounded from them.

3. Just tunings. Perfection in tuning is an ignis fatuus which philosophers and musicians have followed since the beginning of time. They have concentrated on tunings largely composed of primary intervals (the 3/2 fifth and the 5/4 third) and which are loosely termed ‘just tunings.’ Complexity, vagueness and the absence of a simple method for recording observations have caused confusion and reduplication, but with a simple definition and a geometrical analogy suggested by T. H. O’Beirne [4] I hope to show that there are 118 just tunings and all possess undesirable qualities in varying degree.
A just tuning is one in which every note is related to at least one other note by a primary interval. Such a tuning can be plotted on squared paper. Vertices represent notes, horizontal lines joining them (left-right) perfect fifths, and vertical lines (up-down) major thirds. The problem resolves itself into finding the total number of unbroken patterns that can be formed.

Patterns are easily memorised and each completely defines a tuning. We start with the simplest—Pythagorean tuning—a sequence of eleven perfect fifths which can be plotted on a single horizontal line (Figure 3, i). The twelfth fifth uniting the last note with an octave of the first is left blank, and this indicates that it is imperfect. It is called the Procrustean fifth since it is cut to fit, and in this instance it is a perfect fifth less a Pythagorean comma: $2^{18}/3^{11}$. See [5].

Next we list all possible patterns occupying two horizontal lines (Figure 3, ii–xii). The symmetrical pattern (vii) with ten perfect fifths and four major thirds was suggested by Ramis de Pareja in 1482.

There are 43 patterns occupying three horizontal lines, and space will not allow these to be listed. The symmetrical pattern (Figure 4) is of special interest. I regard it as the most perfect of all the just tunings because it contains the maximum number of primary intervals (nine perfect fifths, eight major thirds).

Four horizontal lines give 55 patterns. The following (Figure 5) by Marpurg 1776, is generally spoken of as the ‘model form’ of just tuning, although it has one less perfect fifth than the symmetrical pattern above.

Five horizontal lines give 8 patterns and this completes the list of 118 just tunings.
Just tunings are pleasing, and each key has a character which can be suited to the mood of the composition (now a forgotten artistic refinement). Their inherent imperfections render them unacceptable for the harmonic and modulatory demands of modern music.

4. Temperaments. Unpleasant intervals cannot be abolished and improvement is only obtained by “tempering” or adjusting, so that the unpleasantness is shared with other intervals. This can be done in an infinite number of ways of which two will be outlined.

Equal Temperament (ET). The Chinese were concerned with the problem of dividing the octave into twelve equal intervals more than 1000 years B.C. Their music did not require twelve chromatic notes but they realised the need for this number for the purpose of transposition. This meant finding the twelfth root of two which was not an easy problem. The astronomer Ho-Cheng-Tien was accused of “doing violence to figures” when he tried to find a solution ca. 420 A.D. Wang-Po, a physician, produced inaccurate results ca. 938 A.D., and not until 1598 did Prince Chu-Tsai-Yu “after meditating for days and nights before the light of Truth was revealed” come up with an answer said to be correct to nine places. In Europe the same feat was performed in 1600 by Simon Stevin, an inspector of canals in Holland, author of La Disme, and inventor of a sailing barge.

There is no evidence that J. S. Bach (1685–1750) attempted or intended to tune equally. The “48” were written for “Das wohltemperierte Clavier”—the well tempered clavier, not the equally tempered. It has been pointed out that the frets on ancient instruments appear to be spaced equally (in the logarithmic sense) and the 6-string lute in The Ambassadors by Hans Holbein the Younger 1533, has been quoted as an example. This instrument, and a number of curious objects including a German arithmetic book, lie on the lower shelf of the buffet on which the ambassadors are leaning. The finger board is foreshortened by perspective, and all in all the example is not convincing.

Equal temperament was not generally adopted until the beginning of the present century. It is a tedious temperament for the tuner because every interval is “out of tune.” Accuracy is seldom achieved and then only by counting twelve different beat rates or by utilising apparatus such as the “Stroboconn.”

Meantone temperament. In 1523 Aron suggested that fifths should be tempered to produce 5/4 thirds. Four perfect fifths—say C-G-D-A-E—produce a third C-E (plus two octaves) with an unpleasantly large ratio, i.e., \((3/2)^4\)
divided by 4 to get rid of the octaves $= 81/64$. In order to give a $5/4$ third, the ratio of each fifth must be reduced to $\sqrt{5}$.

Meantone tuning is not just, because the network is broken (Figure 6). The middle note of the major third divides the latter into two equal major seconds, hence the name "meantone."

This temperament was established by about 1600 and remained popular for a long time. Many organs were still tuned to it at the beginning of the present century. It died a lingering death because musicians were strongly opposed to its replacement by equal temperament.

5. Equal beating scale (EBS). This is evolved in a different manner and is not a tuning or a temperament. It possesses the following advantages:

i. It can be used for all musical purposes.
ii. It introduces a soupçon of colour to all keys.
iii. It may represent a close approximation to J. S. Bach's "well-tempered" scale.
iv. It enhances the resonance of stringed keyboard instruments.
v. Above all, ease in tuning is marked and greater accuracy is likely to be achieved.

The principle is simple. All intervals in the tuning series (Figure 2) have the same beating rate. Beats occur between the 3rd partial of the lower note and the 2nd partial of the upper note of an imperfect fifth (Figure 7, a): and between the 3rd partial of the upper note and the 4th partial of the lower note of an imperfect fourth (Figure 7, b).

The beat rate is the difference between the frequencies of these partials. In this example, if the frequency of the lower E is half the rate of the upper E (i.e., they are "in tune") then the beat rate is the same in each case.

The common beat rate ($\beta$) for the EBS is found by solving the twelve chain equations of the tuning series (Figure 2) in terms of $\beta$ and $a$. 
Thus

\begin{align*}
(i) \quad & \beta = 3a - 2e \quad \text{or} \quad e = \frac{3}{2}a - \frac{3}{2}b \\
(ii) \quad & \beta = 3e - 4b \quad \text{or} \quad b = 3\frac{e}{2} - \frac{1}{2}b \\
(iii) \quad & \beta = 3b - 2f# \quad f# = \frac{3}{2}b - \frac{1}{2}b \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
(xii) \quad & 2a = \frac{3}{2}d - \frac{1}{2}b.
\end{align*}

From which \( \beta = (7153/1568693)a \). This is the key of all necessary calculations. The ratios of all intervals can be expressed as integral numbers, and with international tuning frequency \( A_4 = 440 \) cps the beat rate is 1.00317 p/s which, for most practical purposes, may be taken as unity.

6. Apologia. It has been said, with regard to musical problems, that musicians generally give the correct answers supported by illogical argument, but mathematicians arrive at incorrect answers through a process of irrefutable reasoning.

This puts me in a quandary. I would like to be thought of as the operator "little i"—neither one thing nor the other, perhaps imaginary but sometimes useful.

References


ONTTO ENDOMORPHISMS ARE ISOMORPHISMS

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1. Introduction. In this paper we shall discuss a theorem of Vasconcelos, and the following generalization thereof:

THEOREM 1. Let \( M \) be a finitely generated \( R \)-module over the commutative ring \( R \). Let \( N \) be any \( R \)-submodule of \( M \). Let \( f: N \to M \) be an \( R \)-module epimorphism. Then \( f \) is an isomorphism.

Vasconcelos's result covers the case when \( N = M \). In [8] it is stated more generally, for modules \( M \) whose localizations \( M_m \) are finitely generated \( R_m \)-modules for each maximal ideal \( m \) of \( R \). But since \( f \) is onto (respectively one-one)